Second-Order Risk of Alternative Risk Parity Strategies

Simone Bernardi*
University of Zurich

Markus Leippold†
University of Zurich

Harald Lohre‡
Invesco
EMP, Lancaster University Management School

March 2, 2018

*Correspondence Information: University of Zurich, Department of Banking and Finance, Plattenstrasse 14, 8032 Zurich, Switzerland; simone.bernardi@uzh.ch

†Correspondence Information (Contact Author): University of Zurich, Department of Banking and Finance, Plattenstrasse 14, 8032 Zurich, Switzerland; markus.leippold@bf.uzh.ch

‡Correspondence Information: Invesco Quantitative Strategies, An der Welle 5, 60322 Frankfurt am Main, Germany and Centre for Financial Econometrics, Asset Markets and Macroeconomic Policy, Lancaster University Management School, Bailrigg, Lancaster LA1 4YX, United Kingdom; harald.lohre@invesco.com

Second-Order Risk of Alternative Risk Parity Strategies

ABSTRACT

The concept of second-order risk operationalizes the estimation risk in portfolio construction induced by model uncertainty. We study its contribution to the realized volatility of recently developed alternative risk parity strategies that invest in an uncorrelated decomposition of the asset universe. For each strategy, we derive closed-form solutions for the second-order risk, subsequently illustrated in empirical analysis based on real market data. The results suggest a relation between the contribution of second-order risk and the sensitivity of a portfolio to single eigenvectors of the covariance matrix of assets' returns. Among the strategies considered, we find the principal risk parity strategy, that invests equally in each eigenvector underlying the variance–covariance matrix, to be immune to second-order risk. For the other strategies, second-order risk can be partially mitigated by means of statistical methods. In particular, we provide evidence for the eigenvalue adjustment being the most effective method for correcting the the SOR bias.

Keywords: Estimation Risk, Second-Order Risk, Portfolio Construction, Risk Parity, Diversification

JEL Classification: G11; D81

I. Introduction

When modern portfolio theory emerged with the seminal paper of Markowitz (1952) on mean–variance (MV) optimization, estimation risk was mostly neglected and the estimated parameters were treated as if they were the true parameters. However, especially in finance, estimation risk is unavoidable. As indicated by a wide number of authors, such as Jobson, Korkie, and Ratti (1979), Jorion (1986), and Michaud (1989), allocating assets following the mean–variance paradigm without recognizing the existence of the estimation risk inherent in the parameters has a huge impact on the optimized portfolios and leads to several undesirable features and deficiencies, such as, unstable weights, concentrated allocations, excessive portfolio turnover, lower returns, and the realized portfolio volatility's exceeding the ex ante expected volatility.[1]

This last deficiency, a higher realized volatility than expected, has been studied in Shepard (2009), who defined a risk measure to quantify what he dubbed the Second-Order Risk (SOR) bias in optimized portfolios, i.e., a systematic deviation, induced by model uncertainty, of realized volatility from in-sample volatility. Sheppard also shows that the unconstrained minimum-variance (MV) portfolio suffers from a systematic SOR bias, which is proportional to the ratio of the number of assets to the number of in-sample observations considered in the portfolio construction. The analyzed SOR bias is consequently found to be especially pronounced for short estimation periods. Only when the number of observations is considerably greater than the number of assets does the SOR bias tend to disappear. More recently, Stefanovits, Schubiger, and Wütrich (2015) extend the empirical study of the SOR bias to the most-diversified portfolio approach of Choueifaty and Coignard (2008) and to the traditional risk parity of Maillard, Roncalli, and Teiletche (2010), with similar findings, suggesting that the SOR bias is rather a common denominator across different portfolio construction methods than an undesirable side effect limited to the MV optimization framework.

In the present paper we advance the study of SOR bias by looking at its contribution to the realized volatility of recently developed alternative risk parity strategies that allocate along Principal Portfolios (PPs), i.e., the eigenvectors out of a principal component analysis (PCA)

[1]See, for example, DeMiguel, Garlappi, and Uppal (2009), who empirically corroborate that some 500 years of monthly in-sample observations are necessary for the MV portfolio to outperform the $1/N$ portfolio in an asset universe of 50 U.S. stocks.

of the covariance matrix of the assets returns. Inspired by the work of Partovi and Caputo (2004), various authors have recognized the appealing properties of investing in terms of principal portfolios which have zero correlations by design, unlike the correlations of the underlying assets. Hence, PPs allow for a more natural description of the diversification properties of the asset universe, see Meucci (2009).

Recently, the Diversified Risk Parity (DRP) strategy, the analogue of the risk parity strategy of Maillard, Roncalli, and Teiletche (2010) but in the principal portfolio space, aims at equally weighting the contribution of each principal portfolio to the risk, where the risk is measured in terms of the volatilities of the PPs. The DRP strategy has been studied in Lohre, Neugebauer, and Zimmer (2012) with a focus on U.S. equities, and in Kind (2013) as well as Lohre, Opfer, and Ország (2014) in a multi-asset investment universe. In its optimal unconstrained version, the DRP strategy invests in each PP proportionally to the inverse of the square root of the corresponding eigenvalues, i.e., proportional to the inverse of the volatilities of the PPs. In addition, we consider two alternative variations of the DRP strategy. First, we look at the $1/V$ portfolio, which weights the PPs by the inverse of the corresponding eigenvalues, i.e., proportional to the inverse of the variances of the PPs. Second, we consider a strategy mentioned in Hall (2012). The author proposes a strategy which could be interpreted as the analogue of the equally weighted portfolio, or $1/N$ strategy, in PP space. This strategy suggests investing in the main uncorrelated risk sources in a more natural fashion than the DRP and the $1/V$ portfolio. Instead of having equal budget risks across PPs, the risk is budgeted proportionally to the contribution of each PP to the total variance. As a result, the lion's share of capital is allocated to the most significant uncorrelated risk source, that potentially carry risk premia with a higher probability, hence basically neglecting most of the less significant PPs. Technically, one is simply allocating equal weights to the PPs, which is why we call it Principal Risk Parity (PRP).

The common denominator of the considered alternative risk parity strategies is that they invest in an uncorrelated decomposition of the asset universe provided by the PCA. We note that the PCA is by no means unique in decomposing the asset universe into uncorrelated risk sources. Leveraging on the work of Meucci, Santangelo, and Deguest (2015), various authors have proposed more sophisticated versions of alternative risk parity strategies, which we do not consider in this study. Among others, Kind and Poonia (2015) apply the minimum torsion directly to the underlying assets. Bernardi, Leippold, and Lohre (2018) start from an economically well founded commodity factor model and obtain uncorrelated risk sources by applying minimum rotations to it.

For each selected alternative risk parity strategy we derive analytical closed-form solutions for the corresponding SOR bias, assuming the assets' returns are normally distributed. The SOR bias of the $1/V$ strategy (which invests in the PPs in inverse proportion to their variance) is comparable to the SOR bias of the MV portfolio. As for the optimal DRP strategy (which instead invests in the PPs in proportion to the square root of the inverse of the corresponding principal portfolio's variances) we show that its SOR bias is approximately equal to the square root of the SOR bias of the $1/V$ and MV portfolios. This correlation between the magnitude of the corresponding SOR bias and the weight assigned to the PPs sheds light on the lower exposure of the DRP strategy to low-volatility PPs. This observation enhances expectations about an even lower SOR bias for the PRP strategy (as this portfolio invests in the PPs by equally weighting them, i.e., by allowing every single PP to contribute to the overall portfolio volatility proportionally to its own volatility). Also, the PRP strategy appears to be immune from the SOR bias and therefore not subject to systematic risk underestimation. All in all, these results shed an interesting light on the relation between the weighting of the principal portfolios and the contribution of SOR to a realized portfolio's volatility. These findings resonate well with the observation in DeMiguel, Garlappi, and Uppal (2009), who report less sensitivity to estimation risk for the correlation matrix than the covariance matrix when these are used as inputs to portfolio optimization. By equally weighting the PPs, the PRP strategy exclusively leverages the estimation of the correlation structure of the asset returns as given by the corresponding loading matrix defining the PPs. Instead, the $1/V$, DRP, and MV strategies require the additional estimation of the eigenvalues for their portfolio optimization.

Our analytical results are further confirmed by empirical analysis. We consider alternative risk parity strategies (together with the MV, the equally weighted, and a random portfolio as controls) and calculate their SOR bias over the last 10 years by running these strategies on the Fama–French industry portfolios of real equity market data as provided by the Center for Research in Security Prices (CRSP). We assess the SOR contribution to realized volatility by varying the number of assets as well as the number of in-sample observations considered. Additionally, we compare the SOR bias before and after applying estimation risk mitigation procedures to the covariance matrix. In particular, we look at simple bootstrapping, the linear shrinkage of Ledoit and Wolf (2004a), and the eigenvalue adjustment of Menchero, Wang, and Orr (2012).[2] In contrast to eigenvalue adjustment, bootstrapping historical returns and linear shrinkage do not directly address the SOR bias in portfolio construction. Still, both methods seek to mitigate the negative effects of estimation risk. Given that the SOR bias is a measure of the estimation risk as well, we expect these methods to positively influence that dimension. In our empirical study we find that linear shrinkage only marginally helps to reduce the SOR bias. The simple bootstrapping method hardly mitigates the SOR bias across the analyzed strategies.

The rest of this paper is organized as follows. Section II outlines the theoretical framework and provides closed-form solutions for the SOR bias of alternative risk parity strategies. Section III provides empirical evidence of the effects of SOR on the realized volatility of portfolio construction strategies and sheds light on the performance of well-known estimation risk-mitigation methodologies on the SOR bias. Section IV concludes the paper.

II. Second-Order Risk and Risk-Based Portfolio Construction

A. Second-Order Risk

In this section we describe the framework we use to assess the estimation risk in portfolio construction strategies. Due to the difficulties in predicting returns, our study focuses on the effects of estimation risk for a set of risk-based portfolio construction strategies which rely on the sample covariance matrix of the assets' excess returns as the sole parameter. We consider a sample period of length $T \in \mathbb{N}$, (e.g., T trading days) and a realized (or out-of-sample) period of the same length. Going forward we will differentiate between statistics constructed for the in-sample and those constructed for the realized period by labeling them accordingly. In this setup, we consider

[2] Among others, Johnstone (2001), Ledoit and Wolf (2004b), El Karoui (2008), and Stefanovits, Schubiger, and Wütrich (2015) study the spectrum of the sample covariance matrix and propose alternative estimators of the covariance matrix, derived by individual adjustment of each eigenvalue. Although the sample estimate of the covariance matrix is an unbiased estimator of the true covariance matrix, these authors demonstrate that its eigenvalues greatly deviate from the true ones, especially when the number of assets considered greatly exceed the sample size. There is a systematic upward bias for the largest sample eigenvalues, whereas the smallest ones appear to be slightly biased downwards.

a universe of N assets with sample excess returns $\hat{\mathbf{R}} \in \mathbb{R}^{N\times T}$ and covariance matrix $\hat{\mathbf{\Omega}} \in \mathbb{R}^{N\times N}$, where

$$\hat{\mathbf{\Omega}} = \frac{1}{T}\hat{\mathbf{R}}\hat{\mathbf{R}}^{'}. \tag{1}$$

Analogously, the realized excess return is denoted by $\mathbf{R} \in \mathbb{R}^{N\times T}$ and the covariance matrix by $\mathbf{\Omega} \in \mathbb{R}^{N\times N}$, where

$$\mathbf{\Omega} = \frac{1}{T}\mathbf{R}\mathbf{R}^{'}. \tag{2}$$

Then, we define $\hat{\mathbf{w}} := w(\hat{\mathbf{R}}, \hat{\mathbf{\Omega}})$ to be a vector of portfolio weights derived on the basis of sample returns and volatility matrix. In our framework, the estimation risk is measured by looking at the second-order risk of portfolio strategies, defined as the ratio of realized over sample portfolio variance. Hence, for a portfolio strategy $\hat{\mathbf{w}}$, the corresponding second-order risk bias can be computed as follows

$$SOR(\hat{\mathbf{w}}) := \frac{\sigma^2_{realized}(\hat{\mathbf{w}})}{\sigma^2_{sample}(\hat{\mathbf{w}})} = \frac{\mathbb{E}\left[\hat{\mathbf{w}}^{'}\mathbf{\Omega}\hat{\mathbf{w}}\right]}{\mathbb{E}\left[\hat{\mathbf{w}}^{'}\hat{\mathbf{\Omega}}\hat{\mathbf{w}}\right]}. \tag{3}$$

On the one hand, an SOR bias equal to 1 characterizes a robust portfolio construction strategy. This is trivially the case for the $1/N$ portfolio, whose weights are proportional to

$$\mathbf{w}_{1/N} := \mathbf{1}. \tag{4}$$

Obviously, these weights do not depend on the sample covariance matrix. The SOR bias for this portfolio is derived as follows

$$SOR(\mathbf{w}_{1/N}) = \frac{\mathbb{E}\left[\mathbf{1}^{'}\mathbf{\Omega}\mathbf{1}\right]}{\mathbb{E}\left[\mathbf{1}^{'}\hat{\mathbf{\Omega}}\mathbf{1}\right]} = \frac{\mathbf{1}^{'}\mathbf{\Omega}\mathbf{1}}{\mathbf{1}^{'}\mathbb{E}\left[\hat{\mathbf{\Omega}}\right]\mathbf{1}} = 1. \tag{5}$$

On the other hand, an SOR bias different from 1 indicates that the portfolio strategy systematically suffers from estimation risk; this calls for the application of estimation risk mitigation techniques.

Another popular example is the minimum-variance portfolio (MV). Shepard (2009) shows that the MV portfolio with weights proportional to

$$\hat{\mathbf{w}}_{MV} := \hat{\mathbf{\Omega}}^{-1}\mathbf{1}, \tag{6}$$

suffers from a systematic SOR bias, which is proportional to the ratio of the number of assets N to the length of the sample period T. More precisely, under the assumption that the underlying asset excess returns are normally distributed with mean zero and covariance matrix $\mathbf{\Omega}$, the sample covariance matrix $\hat{\mathbf{\Omega}}$, as defined in equation (1), follows a Wishart distribution, i.e., $T \times \hat{\mathbf{\Omega}} \sim \mathcal{W}_N(\mathbf{\Omega}, T)$.[3] Using this result, (Shepard 2009) derives a closed-form formula for the SOR bias of the optimal MV portfolio $\hat{\mathbf{w}}_{MV}$ given by equation (6) as

$$\begin{aligned} SOR(\hat{\mathbf{w}}_{MV}) &= \frac{\mathbb{E}\left[\hat{\mathbf{w}}_{MV}'\mathbf{\Omega}\hat{\mathbf{w}}_{MV}\right]}{\mathbb{E}\left[\hat{\mathbf{w}}_{MV}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}_{MV}\right]} = \frac{\mathbf{1}'\mathbb{E}\left[\hat{\mathbf{\Omega}}^{-1}\mathbf{\Omega}\hat{\mathbf{\Omega}}^{-1}\right]\mathbf{1}}{\mathbf{1}'\mathbb{E}\left[\hat{\mathbf{\Omega}}^{-1}\hat{\mathbf{\Omega}}\hat{\mathbf{\Omega}}^{-1}\right]\mathbf{1}} \\ &\overset{(A)}{\simeq} \left(1-\frac{N}{T}\right)^{-3}\frac{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}}{\mathbf{1}'\mathbb{E}\left[\hat{\mathbf{\Omega}}^{-1}\right]\mathbf{1}} \\ &\overset{(B)}{\simeq} \left(1-\frac{N}{T}\right)^{-2}\frac{\mathbf{1}'\mathbb{E}\left[\hat{\mathbf{\Omega}}^{-1}\right]\mathbf{1}}{\mathbf{1}'\mathbb{E}\left[\hat{\mathbf{\Omega}}^{-1}\right]\mathbf{1}} \\ &= \left(1-\frac{N}{T}\right)^{-2}, \end{aligned}$$

where approximations (A) and (B) are based on results regarding the first two moments of the inverse Wishart distribution which drop $\mathcal{O}(1/N)$ and $\mathcal{O}(1/T)$ terms for simplicity.[4]

Not only does Shepard (2009) quantify the second-order risk induced by the weights of the MV portfolio, he also uses the above result to obtain an unbiased estimator for its variance:

$$\hat{\sigma}_{MV}^2 := \hat{\mathbf{w}}_{MV}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}_{MV}\left(1-\frac{N}{T}\right)^{-2}, \tag{7}$$

and it thus holds that

$$\mathbb{E}\left[\hat{\sigma}_{MV}^2 \middle| \mathbf{\Omega}\right] = \hat{\sigma}_{MV}^2. \tag{8}$$

[3] The Wishart distribution is the multivariate case of the χ^2-distribution.

[4] In particular, for the Inverse Wishart distribution, one has

$$(A) \qquad \mathbb{E}\left[\hat{\Omega}^{-1}\Omega\hat{\Omega}^{-1}\right] = \frac{(T-1)T^2}{(T-N)(T-N-1)(T-N-3)}\Omega^{-1} \simeq \left(1-\frac{N}{T}\right)^{-3}\Omega^{-1} \overset{(B)}{\simeq} \left(1-\frac{N}{T}\right)^{-2}\mathbb{E}\left[\hat{\Omega}^{-1}\right]$$

$$(B) \qquad \mathbb{E}\left[\hat{\Omega}^{-1}\right] = \left(1-\frac{N-1}{T}\right)^{-1}\Omega^{-1} \simeq \left(1-\frac{N}{T}\right)^{-1}\Omega^{-1}.$$

This correction depends only on the number of assets and the number of observations used for the in-sample calculation of the covariance matrix. For example, assuming an in-sample period of 1 year (with approximately 252 trading days) and an asset universe of 75 securities, equation (7) implies that the realized out-of-sample variance of the MV portfolio returns is twice as high as the in-sample predicted portfolio variance.

B. Second-Order Risk of Alternative Risk Parity Strategies

In the following we examine the SOR bias pertaining to alternative risk parity strategies that invest in uncorrelated risk sources embedded in the underlying asset universe. These uncorrelated risk sources are the principal components (or principal portfolios, PP) of the PCA decomposition of the sample covariance matrix, i.e.,

$$\hat{\mathbf{\Omega}} = \hat{\mathbf{U}}' \hat{\mathbf{\Lambda}} \hat{\mathbf{U}}, \tag{9}$$

where $\hat{\mathbf{U}}$ is the matrix of eigenvectors of $\hat{\mathbf{\Omega}}$ representing the loadings of the principal components and $\hat{\mathbf{\Lambda}} = \text{diag}(\lambda_i)_{i=1,\ldots,N}$ is the diagonal matrix containing the variances of the corresponding eigenvalues. Because of the uncorrelatedness of the PPs, their marginal contribution to portfolio diversification is considerable. The overall portfolio variance σ^2_{sample} can be represented as the weighted sum of the variances of the PPs:

$$\sigma^2_{sample} := \hat{\mathbf{w}}' \hat{\mathbf{\Omega}} \hat{\mathbf{w}} = \sum_{i=1}^{N} \tilde{w}_i^2 \hat{\lambda}_i, \tag{10}$$

where $\tilde{\mathbf{w}} := \hat{\mathbf{U}} \hat{\mathbf{w}}$ translates portfolio weights $\hat{\mathbf{w}}$ into principal portfolio weights $\tilde{\mathbf{w}}$. Then, a portfolio's SOR bias is the weighted average (based on the weights of the PP) of the SOR bias of each single PP. Systematically altering the weighting of the PPs can thus be expected to lead to a systematic change in the SOR bias of the portfolio. We investigate this aspect by looking at the related portfolio strategies.

B.1. Inverse-variance in principal portfolios

First, we consider the $1/V$ portfolio that invests in each PP proportionally to the inverse of the corresponding PP's variance[5] and thus strongly loads on low volatility PPs. The portfolio weights are proportional to

$$\hat{\mathbf{w}}_{1/V} := \hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1}\mathbf{1}. \tag{11}$$

To compute the SOR bias we consider the unnormalized version of portfolio weights in equation (11) and separately evaluate the realized and the sample risk estimates. For the former, we obtain

$$\begin{aligned}
\mathbb{E}\left[\hat{\mathbf{w}}_{1/V}'\mathbf{\Omega}\hat{\mathbf{w}}_{1/V}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}'\hat{\mathbf{\Lambda}}^{-1}\hat{\mathbf{U}}'\mathbf{\Omega}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1}\mathbf{1}\middle|\hat{\mathbf{U}}\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}'\hat{\mathbf{U}}'\mathbb{E}\left[\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1}\hat{\mathbf{U}}'\mathbf{\Omega}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1}\hat{\mathbf{U}}'\middle|\hat{\mathbf{U}}\right]\hat{\mathbf{U}}\mathbf{1}\right] \\
&= \mathbb{E}\left[\mathbf{1}'\hat{\mathbf{U}}'\mathbb{E}\left[\hat{\mathbf{\Omega}}^{-1}\mathbf{\Omega}\hat{\mathbf{\Omega}}^{-1}\middle|\hat{\mathbf{U}}\right]\hat{\mathbf{U}}\mathbf{1}\right] \\
&\overset{(A)}{\simeq} \left(1-\frac{N}{T}\right)^{-2}\mathbb{E}\left[\mathbf{1}'\hat{\mathbf{U}}'\hat{\mathbf{\Omega}}^{-1}\hat{\mathbf{U}}\mathbf{1}\right] \\
&= \left(1-\frac{N}{T}\right)^{-2}\sum_{k=1}^{N}\mathbb{E}\left[1/\hat{\lambda}_k\right].
\end{aligned}$$

For the sample risk estimate, we obtain

$$\mathbb{E}\left[\hat{\mathbf{w}}_{1/V}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}_{1/V}\right] = \mathbb{E}\left[\mathbf{1}'\hat{\mathbf{\Lambda}}^{-1}\hat{\mathbf{U}}'\hat{\mathbf{\Omega}}\hat{\mathbf{U}}\hat{\mathbf{\Lambda}}^{-1}\mathbf{1}\right] = \mathbb{E}\left[\mathbf{1}'\hat{\mathbf{\Lambda}}^{-1}\hat{\mathbf{\Lambda}}\hat{\mathbf{\Lambda}}^{-1}\mathbf{1}\right] = \sum_{k=1}^{N}\mathbb{E}\left[1/\hat{\lambda}_k\right], \tag{12}$$

Hence, the SOR bias of the $1/V$ portfolio can be approximated as follows:

$$SOR(\hat{\mathbf{w}}_{1/V}) = \frac{\mathbb{E}\left[\hat{\mathbf{w}}_{1/V}'\mathbf{\Omega}\hat{\mathbf{w}}_{1/V}\right]}{\mathbb{E}\left[\hat{\mathbf{w}}_{1/V}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}_{1/V}\right]} \simeq \left(1-\frac{N}{T}\right)^{-2}, \tag{13}$$

The SOR bias derived in equation (13) for the $1/V$ strategy is thus of the same magnitude as the SOR bias of the MV portfolio in equation (7). Thus, equation (7) is an unbiased estimator of the variance of the $1/V$ strategy as well.

[5] As every one of the principal portfolios can be bought or sold, there exist 2^N asset allocations, where N is the number of principal portfolios, all of which are inverse variance strategies. A unique strategy is consequently obtained by imposing a sign constraint on the principal portfolios. For the empirical part of this paper, we align the sign of each principal portfolio with the one of its corresponding historical risk premia over a given time period.

B.2. Diversified Risk Parity: Inverse volatility along principal portfolios

Second, we consider the diversified risk parity (DRP) strategy. Similar to the $1/V$ portfolio, the DRP strategy invests in uncorrelated risk sources as provided by the PPs pertaining to the PCA decomposition of the sample covariance matrix. The DRP strategy especially leverages the following diversification measure of Meucci (2009):

$$\mathcal{N}_{Ent}(\mathbf{w}) = \exp\left(-\sum_{i=1}^{N} p(\tilde{w}_i)\ln p(\tilde{w}_i)\right), \tag{14}$$

where

$$p(\tilde{w}_i) = \frac{\tilde{w}_i^2\hat{\lambda}_i}{\sum_{i=1}^{N}\tilde{w}_i^2\hat{\lambda}_i}, \qquad i = 1, ..., N. \tag{15}$$

$\mathcal{N}_{Ent}(\mathbf{w})$ can be interpreted as the number of uncorrelated risk sources that a given portfolio strategy $\mathbf{w}$ is investing in. One has $\mathcal{N}_{Ent}(\mathbf{w}) = 1$ for a fully concentrated strategy and $\mathcal{N}_{Ent}(\mathbf{w}) = N$ for a fully diversified strategy. The weights of the DRP strategy are constructed by maximizing the diversification measure $\mathcal{N}_{Ent}(\mathbf{w})$, i.e.,

$$\hat{\mathbf{w}}_{DRP} = \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}}\,\mathcal{N}_{Ent}(\mathbf{w}). \tag{16}$$

Maximizing the diversification is equivalent to allocating an equal risk budget to every uncorrelated PP, subject to a set of allocation constraints $\mathcal{C}$. In the absence of constraints, the DRP strategy has a closed-form solution that prescribes inverse volatility investing along principal portfolios.[6] Its weights are proportional to

$$\hat{\mathbf{w}}_{DRP} := \hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-\mathbf{1/2}}\mathbf{1}. \tag{17}$$

By weighting each PP inversely to the square root of its variance, the DRP ensures an equal contribution to the total variance by each PP, reflecting the underlying idea of maximum diversification. The DRP weighting appears to be more moderate, when compared to that of the $1/V$ strategy. This more moderate weighting of the PPs should translate into a lower SOR bias.

[6] As with the $1/V$ strategy, uniqueness of the asset allocation is again guaranteed by means of sign constraints based on the historically observed risk premia.

Similar to the derivation of the SOR bias for the $1/V$ portfolio, we calculate the realized risk as well as the sample risk estimates separately. For the realized risk, we have

$$\begin{aligned}
\mathbb{E}\left[\hat{\mathbf{w}}_{DRP}^{'}\boldsymbol{\Omega}\hat{\mathbf{w}}_{DRP}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}^{'}\hat{\boldsymbol{\Lambda}}^{-1/2}\hat{\mathbf{U}}^{'}\boldsymbol{\Omega}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-1/2}\mathbf{1}\middle|\hat{\mathbf{U}}\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}^{'}\hat{\mathbf{U}}^{'}\mathbb{E}\left[\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-1/2}\hat{\mathbf{U}}^{'}\boldsymbol{\Omega}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-1/2}\hat{\mathbf{U}}^{'}\middle|\hat{\mathbf{U}}\right]\hat{\mathbf{U}}\mathbf{1}\right] \\
&= \mathbb{E}\left[\mathbf{1}^{'}\hat{\mathbf{U}}^{'}\mathbb{E}\left[\hat{\boldsymbol{\Omega}}^{-1/2}\boldsymbol{\Omega}\hat{\boldsymbol{\Omega}}^{-1/2}\middle|\hat{\mathbf{U}}\right]\hat{\mathbf{U}}\mathbf{1}\right] \\
&\overset{(C)}{\simeq} \left(1-\frac{N}{T}\right)^{-1}\mathbb{E}\left[\mathbf{1}^{'}\hat{\mathbf{U}}^{'}\hat{\mathbf{U}}\mathbf{1}\right] = \left(1-\frac{N}{T}\right)^{-1}N,
\end{aligned}$$

where equation (C) follows from a heuristic derivation. The expression

$$\mathbb{E}\left[\hat{\boldsymbol{\Omega}}^{-1/2}\boldsymbol{\Omega}\hat{\boldsymbol{\Omega}}^{-1/2}\middle|\hat{\mathbf{U}}\right],$$

dubbed the "Bias Matrix," cannot be derived analytically. Thus, we verify the validity of our approximation (C) via a Monte Carlo simulation, which we outline in an appendix.

The sample variance estimate can be calculated as follows:

$$\begin{aligned}
\mathbb{E}\left[\hat{\mathbf{w}}_{DRP}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{w}}_{DRP}\right] &= \mathbb{E}\left[\mathbf{1}^{'}\hat{\boldsymbol{\Lambda}}^{-1/2}\hat{\mathbf{U}}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-\mathbf{1/2}}\mathbf{1}\right] \\
&= \mathbb{E}\left[\mathbf{1}^{'}\hat{\boldsymbol{\Lambda}}^{-1/2}\hat{\mathbf{U}}^{'}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^{'}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}^{-\mathbf{1/2}}\mathbf{1}\right] \\
&= \mathbb{E}\left[1^{'}\hat{\boldsymbol{\Lambda}}^{-1/2}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Lambda}}^{-1/2}1\right] = N.
\end{aligned}$$

The SOR bias for the DRP can be approximated as follows:

$$SOR(\hat{w}_{DRP}) := \frac{\mathbb{E}\left[\hat{\mathbf{w}}_{DRP}^{'}\boldsymbol{\Omega}\hat{\mathbf{w}}_{DRP}\right]}{\mathbb{E}\left[\hat{\mathbf{w}}_{DRP}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{w}}_{DRP}\right]} \overset{(C)}{\simeq} \left(1-\frac{N}{T}\right)^{-1}. \tag{18}$$

Analogously to equation (7), we can derive an unbiased estimator of the DRP portfolio's variance:

$$\hat{\sigma}_{DRP}^{2} := \hat{\mathbf{w}}_{DRP}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{w}}_{DRP}\left(1-\frac{N}{T}\right)^{-1}. \tag{19}$$

As in the cases of the MV and $1/V$ portfolios, this correction depends only on the number of assets and the number of observations used for the in-sample calculation of the covariance matrix.

For example, assuming an in-sample period of 1 year (with approximately 252 trading days) and an asset universe of 125 securities, equation (19) implies that the realized out-of-sample portfolio variance is twice as high as the in-sample predicted portfolio variance.

B.3. Principal Risk Parity: Equally-weighted principal portfolios

In light of the above results, we present an alternative risk parity strategy, referred to as principal risk parity (PRP). Similar to the $1/V$ and DRP portfolios, the PRP strategy invests in uncorrelated risk sources as given by the PPs, but in a more natural fashion. The PRP portfolio budgets the risk proportionally to each PP's contribution to the total variance. As a result, the lion's share of the capital is allocated to the most significant principal portfolios, thus getting around the less significant PPs. Technically, we are simply assigning equal weights to the PPs, prompting us to call this strategy the principal risk parity strategy.[7] To obtain the strategy weights $\hat{\mathbf{w}}_{PRP}$, we need to solve

$$\hat{\mathbf{w}}_{PRP} = \underset{\mathbf{w}\in\mathcal{C}}{\operatorname{argmax}}\, \mathcal{M}_{Ent}(\mathbf{w}), \tag{20}$$

where $\mathcal{M}_{Ent}$ is defined as

$$\mathcal{M}_{Ent}(\mathbf{w}) = \exp\left(-\sum_{i=1}^{N} q(\tilde{w}_i)\ln q(\tilde{w}_i)\right), \tag{21}$$

and

$$q(\tilde{w}_i) = \frac{\tilde{w}_i^2}{\sum_{i=1}^{N}\hat{w}_i^2}, \qquad i = 1, ..., N. \tag{22}$$

The unconstrained version has a closed-form solution, proportional to

$$\hat{\mathbf{w}}_{PRP} := \hat{\mathbf{U}}\mathbf{1}. \tag{23}$$

[7] As with the $1/V$ and DRP strategies, uniqueness of the asset allocation is guaranteed by means of sign constraints based on the historically observed risk premia of the PPs.

Intuition suggest that the SOR bias should be lower for the PRP than for the $1/V$, MV, and DRP portfolios, because PRP allocates away from low volatility PPs. Our calculations for PRP confirm this intuition. For the realized variance of the PRP portfolio, we obtain

$$\mathbb{E}\left[\hat{\mathbf{w}}_{PRP}^{'}\boldsymbol{\Omega}\hat{\mathbf{w}}_{PRP}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}^{'}\hat{\mathbf{U}}^{'}\boldsymbol{\Omega}\hat{\mathbf{U}}\mathbf{1}\middle|\hat{\mathbf{U}}\right]\right] \overset{(A)}{=} \mathbb{E}\left[\mathbf{1}^{'}\mathbb{E}\left[\hat{\mathbf{U}}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{U}}\middle|\hat{\mathbf{U}}\right]1\right] = \mathbb{E}\left[\mathbf{1}^{'}\hat{\boldsymbol{\Lambda}}\mathbf{1}\right], \tag{24}$$

where equation (A) relies on the fact that, conditional on $\hat{\mathbf{U}}$:

$$T \times \hat{\boldsymbol{\Omega}} \sim \mathcal{W}_N(\boldsymbol{\Omega}, T) \Rightarrow T \times \hat{\mathbf{U}}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{U}}^{'} \sim \mathcal{W}_N\left(\hat{\mathbf{U}}\boldsymbol{\Omega}\hat{\mathbf{U}}^{'}, T\right). \tag{25}$$

Thus,

$$\mathbb{E}\left[\hat{\mathbf{U}}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{U}}\middle|\hat{\mathbf{U}}\right] = \hat{\mathbf{U}}^{'}\boldsymbol{\Omega}\hat{\mathbf{U}}. \tag{26}$$

The sample variance can be calculated as

$$\mathbb{E}\left[\hat{\mathbf{w}}_{PRP}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{w}}_{PRP}\right] = \mathbb{E}\left[\mathbf{1}^{'}\hat{\mathbf{U}}^{'}\hat{\mathbf{U}}\hat{\boldsymbol{\Lambda}}\hat{\mathbf{U}}^{'}\hat{\mathbf{U}}\mathbf{1}\right] = \mathbb{E}\left[\mathbf{1}^{'}\hat{\boldsymbol{\Lambda}}\mathbf{1}\right]. \tag{27}$$

Hence, the SOR bias of the PRP portfolio is

$$SOR(\hat{\mathbf{w}}_{PRP}) = \frac{\mathbb{E}\left[\hat{\mathbf{w}}_{PRP}^{'}\boldsymbol{\Omega}\hat{\mathbf{w}}_{PRP}\right]}{\mathbb{E}\left[\hat{\mathbf{w}}_{PRP}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{w}}_{PRP}\right]} = 1. \tag{28}$$

In contrast to equation (7) for the MV portfolio, and equation (19) for the DRP portfolio, the sample variance of the PRP strategy can be considered to be an unbiased estimator of its true variance, conditional on correctly estimating the correlation structure of the investment universe:

$$\hat{\sigma}_{PRP}^2 := \hat{\mathbf{w}}_{PRP}^{'}\hat{\boldsymbol{\Omega}}\hat{\mathbf{w}}_{PRP}. \tag{29}$$

C. Mitigating estimation risk

C.1. Eigenvalue adjustment

Not knowing the exact size of the SOR bias of a given portfolio strategy, Menchero, Wang, and Orr (2012) developed an SOR-unbiased estimator of the covariance matrix. Their method is based on the observation that each of the principal portfolios suffers from an SOR bias. In particular, the SOR bias of a given PP is higher if its associated eigenvalue is smaller, see Table I, Panel B.

[Table I about here.]

Their method aims at eliminating the SOR bias of a given portfolio by correcting the SOR bias of the underlying PPs. In this regard, one simulates asset returns that follow a joint normal distribution with mean zero according to the sample covariance matrix. For each simulation $s = 1, .., S$, the authors compute the variances $\lambda_{k,s}$ of the PPs for $k = 1, .., N$ via a PCA of the covariance matrix $\mathbf{\Omega_s}$

$$\mathbf{\Lambda_s} := \mathbf{U}_s^{'}\mathbf{\Omega}_s\mathbf{U}_s. \tag{30}$$

These variances are put into relation with the diagonal elements of a matrix $\tilde{\mathbf{\Lambda}}_s$, which is obtained assuming the sample covariance matrix is the true covariance matrix of the assets' returns, i.e.,

$$\tilde{\mathbf{\Lambda}}_\mathbf{s} = \mathbf{U}_s^{'}\hat{\mathbf{\Omega}}\mathbf{U}_s. \tag{31}$$

Averaging over all simulations gives a simulated SOR bias β_k for every single PP k,

$$\beta_k := \frac{1}{S}\sum_{s=1}^{S}\sqrt{\frac{\tilde{\lambda}_{k,s}}{\lambda_{k,s}}}. \tag{32}$$

For every PP, the sample variance (or eigenvalue) is multiplied by the square of β_k, to build the diagonal elements $\lambda_k^{eig-adj}$, for $k = 1, .., N$ of a matrix $\mathbf{\Lambda}^{eig-adj}$ containing SOR-corrected variances

$$\lambda_k^{eig-adj} := \beta_k^2\lambda_k. \tag{33}$$

The ensuing matrix $\mathbf{\Lambda}^{eig-adj}$ then replaces the diagonal matrix containing the sample eigenvalues in the PCA decomposition to give an SOR-corrected estimator $\hat{\mathbf{\Omega}}^{eig-adj}$ of the covariance matrix of the assets' returns:

$$\mathbf{\Omega}^{eig-adj} = \hat{\mathbf{U}}\mathbf{\Lambda}^{eig-adj}\hat{\mathbf{U}}^{'}. \tag{34}$$

C.2. Bootstrapping

Another simulation-based technique for mitigating estimation risk can be easily obtained by bootstrapping the asset returns instead of their corresponding eigenvectors. Portfolio optimization techniques might be confounded by extreme values in the estimates of the covariance matrix, driven by a low number of observations in the sample period. This problem might be avoided by bootstrapping the asset returns from the sample period. Thus, the simulated covariance matrices $\mathbf{\Omega_s}$, for $s = 1, .., S$, can be used to compute optimal portfolio weights $\mathbf{w}_s = f(\mathbf{\Omega_s})$ and finally—given a sufficiently large number of simulations—the average portfolio weights across simulations might prove to be more robust to estimation risk.

$$\mathbf{w}^{bootstrap} = \frac{1}{S}\sum_{s=1}^{S}\mathbf{w}_s. \tag{35}$$

This simple procedure aims at avoiding extreme and highly volatile asset allocations.

C.3. Linear shrinkage

Lastly, a popular technique for mitigating estimation risk is the linear shrinkage estimator of Ledoit and Wolf (2004a). The authors argue that extreme values in the variance–covariance matrix of asset returns might often shift the optimal weights towards corner solutions. However, such portfolios often disappoint out-of-sample. They suggest shrinking the sample covariance matrix of asset returns $\hat{\mathbf{\Omega}}$ towards a simple estimate of the covariance matrix $\bar{\mathbf{\Omega}}$ consisting of one asset variance σ and one asset covariance δ, i.e., $\bar{\omega}_{i,j} = \sigma$, for $i = j$ and $\bar{\omega}_{i,j} = \delta$, for $i \neq j$. In particular, they derive an estimator as a convex combination of the two, i.e.,

$$\mathbf{\Omega}^{lin-shr} = \tau\bar{\mathbf{\Omega}} + (1-\tau)\hat{\mathbf{\Omega}}, \tag{36}$$

where the weighting factor $0 \leq \tau \leq 1$ is determined via optimization. The use of average values for the variance and covariance components of the simplified covariance matrix, together with the shrinkage procedure, ensures that the linear shrinkage estimator exhibits more moderate values than would have been the case for the sample covariance matrix estimator, thus mitigating the estimation risk.

III. Results

The aim of this section is to illustrate and confirm the results derived in Section II based on empirical analyses of simulated as well as real market data. We first start with simulated data, as this allows us to assume a "true" covariance matrix of asset returns. By means of Monte Carlo simulation, we will show the effectiveness of the unbiased estimators of a portfolio's variance in equations (7), (19), and (29). Figure 1 shows the results of the Monte Carlo simulation for the Minimum Variance (MV), Diversified Risk Parity (DRP), and Principal Risk Parity (PRP) portfolios. The results have been obtained by constraining each portfolio to be fully invested and to have fixed expected return R. For each target return R, a new sample covariance matrix $\hat{\mathbf{\Omega}}$ is estimated on 50 daily returns for each of 25 assets, simulated from a multivariate normal distribution with fixed mean[8] and covariance matrix $\mathbf{\Omega}$. In order to minimize noise in the simulation, we assume the mean return of each asset to be known, and exclusively focus on the simulation of variances and covariances. For a selection of target returns, various versions of a portfolio's volatility are calculated and compared to each other. In particular, the curve labeled "True Frontier" is calculated assuming perfect knowledge of the covariance matrix, i.e., $\sqrt{\mathbf{w}^{*\prime}\mathbf{\Omega}\mathbf{w}^{*}}$, where $\mathbf{w}^{*}$ represents the portfolio that is optimal in terms of the corresponding optimization rule (i.e., either MV, DRP, or PRP) given perfect knowledge about the covariance matrix of asset returns. The dots labeled "Realized" risk represent the actual risk of the portfolio, i.e., $\sqrt{\hat{\mathbf{w}}'\mathbf{\Omega}\hat{\mathbf{w}}}$. The "Naive forecast" is the in-sample estimated volatility of the portfolio, i.e., $\sqrt{\hat{\mathbf{w}}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}}$. The corrected forecast $\sqrt{\hat{\mathbf{w}}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}}\left(1-\frac{N}{T}\right)^{-1}$ (and $\sqrt{\hat{\mathbf{w}}'\hat{\mathbf{\Omega}}\hat{\mathbf{w}}}\left(1-\frac{N}{T}\right)^{-1/2}$ respectively) is obtained by correcting the naive forecast according to equation (7) for the MV portfolio and equation (19) for the DRP portfolio), whereas for the PRP portfolio, the naive forecast in equation (29) is already unbiased and does not need to be corrected. Panel A shows how the naive forecast of the MV portfolio's volatility appears to be even better than the true efficient frontier, whereas the realized volatility turns out to be rather higher. The same effect (with a lesser magnitude) can be observed in Panel B for the DRP portfolio, whereas for PRP (see Panel C), the naive, or sample, volatility forecast does not underestimate the realized volatility as expected.

[8]In the simulation, we allow for different expected returns for the single assets, and constrain the portfolio expected return to match the corresponding fixed expected return R. As an alternative, one may assume each asset to have exactly the same expected return, thus rendering the use of the expected return constraint in the portfolio optimization obsolete.

[Figure 1 about here.]

In addition to simulated data, it is of interest to empirically test the results of Section II on real market data. The use of real market data for the assessment of second-order risk is not straightforward as it is in the case of simulated data, where the true covariance matrix of the asset returns is known. To facilitate this, we make use of the so called SOR bias statistic to assess the accuracy of a portfolio volatility forecast. This is calculated in the following way. For a set of portfolio weights $\mathbf{w} = (\mathbf{w}_t)_{t=1,...,T}$ the SOR bias statistics for $t = t_0 + 1, ..., T$ is given by

$$B_{t_0+t} := \frac{r_t}{\hat{\sigma}_{t-1}} := \frac{\mathbf{w}'_{t-1}\mathbf{R}^d_t}{\sqrt{\mathbf{w}'_{t-1}\hat{\mathbf{\Omega}}_{t-1}\mathbf{w}_{t-1}}}, \tag{37}$$

where t_0 denotes the length of the in-sample rolling window, $\hat{\mathbf{\Omega}}_{t-1}$ denotes the sample covariance matrix of asset returns, and $\mathbf{w}_{t-1}$ denotes the portfolio weights constructed out of de-meaned in-sample returns $\mathbf{R}^d_{t-t_0}, ..., \mathbf{R}^d_{t-1}$. De-meaned returns are considered for the calculation of the bias statistics. This is done to prevent the measure to suffer from a bias induced by asset returns. As $r_t := \mathbf{w}'_{t-1}\mathbf{R}^d_t$ is the realized de-meaned return of the portfolio at time t, B_{t_0+t} can be considered as a standardized return. Once calculated, the standard deviation of the SOR bias statistic provides a good approximation of the SOR bias of the considered portfolio when looked at in the context of real market data.[9]

As asset universe, we build on the Fama–French industry portfolios as provided by the Center for Research in Security Prices (CRSP).[10] Industry portfolios are constructed by equally weighting a selection of U.S. equities grouped by industry type according to the corresponding SIC codes (Standard Industrial Classification). Table I shows a collection of the main statistics for CRSP data composed of daily returns of U.S. equities grouped within ten Industry Portfolios over ten years (from April 2007 to March 2017). These ten portfolios cover the sectors of: Consumer Non-Durables (NoDur), Consumer Durables (Durbl), Manufacturing (Manuf), Oil, Gas, and Coal Extraction and Products (Enrgy), Business Equipment (HiTech), Telephone and Tele-

[9] As pointed out by the referee, while it is straightforward to demean in-sample returns the same cannot be said for out-of-sample returns. While introducing some form of return forecasting might help alleviating potentially remaining biases in the SOR bias statistic, we refrain from adding this layer of complexity to not introduce other sources of model misspecification.

[10] Datasets of from 5 up to 49 industry portfolios are available online. For more details please visit the website: *http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html*.

vision Transmission (Telcm), Wholesale, Retail and Some Services (Shops), Healthcare, Medical Equipment, and Drugs (Hlth), Utilities (Utils), and Other (Other). The same table also reports statistics for the ten Principal Portfolios, the results of PCA decompositions of the asset universe composed by the ten Industry Portfolios over the ten years considered. From the perspective of the SOR bias, it is interesting to observe how the industry portfolios (constructed by equally weighting a selection of U.S. stocks) exhibit SOR biases close to one (in particular ranging from 1.05 to 1.08), whereas the SOR bias of the Principal Portfolios comes out inversely proportional to their volatility with values close to one for the first three principal portfolios (1.04 for PP1, 1.01 for PP2, and 0.95 for PP3), then increasing to values around 2–3 for the last principal portfolios (1.99 for PP9 and 2.73 for PP10).

Based on this asset universe, Figure 2 reports the SOR bias, calculated as the standard deviation of the bias statistic in equation (37) over the ten-year period (from April 2007 to March 2017) for a set of asset allocation strategies which have been presented in Section II—$1/N$, MV, $1/V$, DRP, and PRP—together with a random strategy[11] that has been used as an additional benchmark to the $1/N$ portfolio.[12] The displayed sample SOR bias statistics are in line with the SOR biases theoretically derived in Section II. The same figure reports SOR bias statistics conditional on the application of risk-mitigating techniques, as described in Section II. In addition to the SOR bias derived by the portfolio optimization based on the sample co-variance matrix estimate, Figure 2 reports the SOR bias of the selected asset allocation strategies when the optimization routine is run on each of the alternative estimators of the covariance matrix, namely, an estimate based on a simple bootstrapping method, the linear shrinkage, and the eigenvalue-adjusted technique.

[Figure 2 about here.]

As expected, the equally weighted portfolio and the random strategy do not appear to suffer from an SOR bias: the reported SOR biases, 1.06 for $1/N$ (and 1.05 for the random strategy) are in line with the SOR bias of the single industry portfolios listed in Table I, and ranging from 1.05

[11] The random strategy is based on a weight vector which is randomly drawn at every time tick in the simulation.

[12] The uniqueness of the $1/V$, DRP, and PRP strategies is guaranteed by imposing a sign constraint on the PPs. These have to carry a positive as well as historically observed risk premium, measured by a rolling time window of 12 months.

to 1.08, and are mainly induced by the finite time horizon considered and the reliance on real market data rather than simply simulated data. In contrast to these two control strategies, the MV portfolios SOR bias is rather high at 1.60 (sample) and 1.56 (bootstrap)—i.e., the realized volatility of the MV portfolio is about 60% higher than the in-sample volatility. The SOR bias drops significantly to 1.49 under linear shrinkage and even more, to 1.06, under the eigenvalue-adjustment method. This observation is in line with the average SOR biases of the single Industry Portfolios. The $1/V$ portfolio displays SOR biases, slightly higher but similar in magnitude to those of the MV portfolio. The $1/V$ portfolio has an SOR bias of 1.71 (sample version) and 1.67 (bootstrapping). The SOR bias drops to 1.56 under linear shrinkage of the sample co-variance matrix and to 1.09 under the eigenvalue adjustment. The DRP strategy is less affected by SOR bias than the MV and $1/V$ strategies. Its SOR bias is approximately the square root of those of the $1/V$ and MV strategies, as derived in Section II. We especially observe an SOR bias of 1.42 for the sample and 1.44 for the bootstrap strategies. Applying linear shrinkage or the eigenvalue adjustment appear to give rise to SOR biases of 1.35 and 1.09, respectively. The PRP portfolio displays an SOR bias of 1.10 in its sample and eigenvalue-adjusted versions. The application of bootstrapping and linear shrinkage leaves the bias almost unchanged, at 1.10. The observation of SOR bias statistics slightly higher than one, reported for the strategies in their eigenvalue-adjusted versions, is mainly related to the use of a finite sample of real market data.

In addition to the SOR bias over the whole of the considered time period, Figure 3 depicts the SOR bias statistics for the analyzed portfolio construction strategies over time. The ranking of the SOR bias inferred from Figure 2 also applies over time. The SOR biases of $1/N$ and random strategies oscillate around the expected value of 1. Two major deviations from 1 can be observed for the SOR bias of the $1/N$ strategy in Panel A, prior to the two equity crises in 2000 and 2008. Panel C (MV) and Panel D ($1/V$) are quite similar over time. Panel E reports the SOR bias over time for the DRP portfolio, which is less volatile when compared with the SOR bias of MV and $1/V$ strategies. Panel F gives the SOR bias of the PRP portfolio, which fluctuates around 1 with a slight upward bias. Also, it is less sensitive to the application of methods to mitigate estimation risk. Similarly, applying methods to mitigate the estimation risk over time is in line with the effects observed on average.

[Figure 3 about here.]

Figures 4 and 5 both illustrate the SOR bias for MV, DRP, and PRP portfolios as a function of the number of in-sample observations T and the number of assets N. The results proposed in Figure 5 are derived based on a fixed number of industry portfolios (ten) by letting the number of observations T vary from 20 up to 60 daily returns, giving T/N ratios of from 2 up to 6. The results displayed are consistent with the theory highlighted in Section II. The MV exhibits the highest SOR bias, which increases with a decreasing number of observations. The DRP has a lower SOR bias than the MV. The PRP is unbiased, independently of the number of in-sample observations considered. For the MV and DRP portfolios, the figure also reports the SOR bias after application of the correction highlighted in equation (7) and equation (19), respectively. These appear close to 1, as expected. Panels A to D report the same analysis and vary in the choice of the estimate for the input covariance matrix to portfolio optimization. Panel A reports the results derived from the sample covariance matrix, Panel B repeats the same analysis averaged over multiple bootstrapped covariance matrices, Panel C shows the SOR bias derived using the linear shrinkage estimator, whereas in Panel D the sample covariance matrix used is modified by the eigenvalue adjustment. From this figure we observe how the effect of risk mitigation is homogeneous and proportional to the SOR bias over the T/N ratio.

[Figure 4 about here.]

Figure 4 is derived similarly to Figure 5 (looking at the same T/N ratios) but with a varying number of Industry Portfolios. CRSP provides alternative groupings of the same U.S. equities in Industry Portfolios at various granularities. Panels A to D in Figure 4 are derived based on CRSP groupings of 5, 10, 30, and 48 Industry Portfolios respectively. The number of Industry Portfolios considered seems to have a limited effect on the resulting SOR bias across strategies. Slightly higher figures are observed for lower numbers of assets and lower numbers of observations, most probably a result of noise rather than a deviation from the theory highlighted in Section II.

[Figure 5 about here.]

IV. Conclusion

In this paper we have provided theoretical and empirical evidence on the contribution of second-order risk to realized volatility for alternative risk parity strategies.[13] In particular, we demonstrate that alternative risk parity strategies, such as diversified and principal risk parity, are significantly less sensitive to second-order risk than the classical minimum variance portfolio. In this regard, an adequate allocation of the risk budget along uncorrelated risk sources mitigates potential SOR biases, e.g., by allocating away from lower eigenvalue portfolios, or by relying more on the correlation structure than on the estimates of the eigenvalues in portfolio construction. Taking this insight to an extreme, we show how the principal risk parity strategy which attaches equal weights to uncorrelated risk sources exhibits low to no SOR bias. Additionally, we provide empirical evidence for the eigenvalue adjustment being the most effective in correcting for the SOR bias.

[13] We leave open the question if and to what extent the underlying asset allocation is affected by the SOR bias. It is a-priori not clear if the SOR bias in the portfolio variance estimator is systematically reflected in the single assets' weights. From our investigations, the SOR bias is mainly driven by the SOR bias in the estimator of the eigenvalues of the covariance matrix of assets' returns. This finding suggests the single assets' weights of the principal risk parity to be unbiased as well (as their determination solely relies on the correlation structure and does not require an estimate of the eigenvalues). We leave this conjecture for further investigations.

References

Bernardi, S., M. Leippold, and H. Lohre, 2018, Maximum diversification strategies along commodity risk factors, *European Financial Management* 24, 53–78.

Choueifaty, Y., and Y. Coignard, 2008, Toward maximum diversification, *Journal of Portfolio Management* 34, 40–51.

DeMiguel, V., L. Garlappi, and R. Uppal, 2009, Optimal versus naive diversification: How inefficient is the $1/n$ portfolio strategy?, *Review of Financial Studies* 22, 1915–1953.

El Karoui, N., 2008, Spectrum estimation for large dimensional covariance matrices using random matrix theory, *Annals of Statistics* 36, 2757–2790.

Hall, J.M., 2012, Principal target risk portfolios: An alternative to risk parity, Working paper, AIMA.

Jobson, J.D., B. Korkie, and V. Ratti, 1979, Improved estimation for Markowitz portfolios using James-Stein type estimators, *Journal of the American Statistical Association* 41, 279–284.

Johnstone, I., 2001, On the distribution of the largest eigenvalue in principal components analysis, *Annals of Statistics* 29, 295–327.

Jorion, P., 1986, Bayes-Stein estimation for portfolio analysis, *Journal of Financial and Quantitative Analysis* 21, 279–292.

Kind, C., 2013, Risk-based allocation of principal portfolios, Working paper, Frankfurt-Trust.

Kind, C., and M. Poonia, 2015, Comparing diversification management strategies, *Global Economy and Finance Journal* 8, 67–81.

Ledoit, O., and M. Wolf, 2004a, Honey, I shrunk the sample covariance matrix, *Journal of Portfolio Management* 30, 110–119.

Ledoit, O., and M. Wolf, 2004b, A well conditioned estimator for large dimensional covariance matrices, *Journal of Multivatiate Analysis* 88, 365–411.

Lohre, H., U. Neugebauer, and C. Zimmer, 2012, Diversified risk parity strategies for equity portfolio selection, *Journal of Investing* 21, 111–128.

Lohre, H., H. Opfer, and G. Ország, 2014, Diversifying risk parity, *Journal of Risk* 16, 53–79.

Maillard, S., T. Roncalli, and J. Teiletche, 2010, The properties of equally weighted risk contribution portfolios, *Journal of Portfolio Management* 36, 60–70.

Markowitz, H.M., 1952, Portfolio selection, *Journal of Finance* 7, 77–91.

Menchero, J., J. Wang, and D. Orr, 2012, Improving risk forecasts for optimized portfolios, *Financial Analysts Journal* 68, 40–50.

Meucci, A., 2009, Managing diversification, *Risk* 22, 74–79.

Meucci, A., A Santangelo, and R. Deguest, 2015, Risk budgeting and diversification based on optimized uncorrelated factors, *Risk* 11, 70–75.

Michaud, R., 1989, The Markowitz optimization enigma: Is optimized optimal?, *Financial Analysts Journal* 45, 31–42.

Partovi, M. H., and M. Caputo, 2004, Principal portfolios: Recasting the efficient frontier, *Economics Bulletin* 7, 1–10.

Shepard, P., 2009, Second order risk, Working paper, MSCI Barra.

Stefanovits, D., U. Schubiger, and V. Wütrich, 2015, Model risk in portfolio optimization, *Risks* 2, 315–348.

Table I
Descriptive Statistics of the Fama French Industry Portfolios

The table lists the descriptive statistics for the ten industry portfolios from CRSP (Center for Research in Security Prices). Industry portfolios are constructed by equally weighting a selection of U.S. equities grouped by industry type according to the corresponding SIC codes (Standard Industrial Classification). Industry sectors are caracterized by a short name. This is reported together with a short description. For more details please visit the website: *http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html*. Panel A gives performance and risk statistics of each industry portfolio. Annualized average return and volatility figures are reported together with the corresponding Sharpe Ratio. Value at risk and expected shortfall are computed at the 95% confidence interval over a one year period. Maximum drawdown is calculated over a one year period as well. Second-order risk (SOR) bias is additionally reported. Panel B reports the same statistics for the principal portfolios (PPs), result of a PCA decomposition of the asset universe composed by the ten industry portfolios. Results are derived from ten years of daily returns ranging from April 2007 to March 2017.

Short Name	Description	Return	Volatility	Sharpe Ratio	Value-at-Risk (at 95% level)	Expected Shortfall (at 95% level)	Maximum Drawdown	SOR Bias
Panel A: Industry Portfolios								
NoDur	Consumer Non-Durables	0.7%	4.3%	0.16	-6.4%	-8.2%	-33.2%	1.06
Durbl	Consumer Durables	0.6%	5.9%	0.10	-9.2%	-11.6%	-42.9%	1.06
Manuf	Manufacturing	0.7%	5.6%	0.12	-8.5%	-10.9%	-39.8%	1.07
Enrgy	Oil, Gas, and Coal Extraction and Products	0.1%	8.7%	0.02	-14.2%	-17.9%	-57.7%	1.06
HiTec	Business Equipment	0.7%	4.7%	0.15	-7.1%	-9.0%	-31.4%	1.06
Telcm	Telephone and Television Transmission	0.7%	5.4%	0.12	-8.2%	-10.5%	-38.4%	1.06
Shops	Wholesale, Retail, and Some Services	0.5%	4.9%	0.11	-7.6%	-9.6%	-37.2%	1.06
Hlth	Healthcare, Medical Equipment, and Drugs	0.8%	4.8%	0.17	-7.1%	-9.1%	-27.8%	1.08
Utils	Utilities	0.5%	4.1%	0.13	-6.2%	-7.9%	-27.0%	1.05
Other	Other	0.6%	4.5%	0.14	-6.8%	-8.7%	-29.6%	1.07
Panel B: Principal Portfolios								
PP1	Principal Portfolio 1	1.7%	15.9%	0.11	-24.5%	-31.1%	-110.7%	1.04
PP2	Principal Portfolio 2	0.1%	4.7%	0.02	-7.8%	-9.7%	-23.1%	1.01
PP3	Principal Portfolio 3	0.0%	2.5%	0.02	-4.0%	-5.1%	-10.3%	0.95
PP4	Principal Portfolio 4	0.3%	2.2%	0.12	-3.8%	-4.8%	-9.4%	1.07
PP5	Principal Portfolio 5	0.1%	1.8%	0.06	-3.1%	-3.8%	-7.9%	1.16
PP6	Principal Portfolio 6	0.0%	1.6%	0.03	-2.6%	-3.3%	-6.3%	1.27
PP7	Principal Portfolio 7	0.2%	1.4%	0.11	-2.2%	-2.8%	-6.8%	1.40
PP8	Principal Portfolio 8	0.0%	1.3%	0.03	-2.1%	-2.6%	-6.5%	1.63
PP9	Principal Portfolio 9	0.0%	1.2%	0.04	-2.0%	-2.5%	-4.2%	1.99
PP10	Principal Portfolio 10	0.1%	1.2%	0.05	-2.0%	-2.4%	-5.5%	2.73

Figure 1. Monte Carlo Simulation: Risk–Return Profile vs. Efficient Frontier
The figure shows the risk return profile of mean variance (MV), diversified risk parity (DRP), and principal risk parity (PRP) portfolios obtained via Monte Carlo simulation. Each panel is dedicated to a single portfolio and shows the true efficient frontier, the naive forecast, the corrected forecast, and the realized risk return profile from each simulation. Results are obtained by simulating 50 daily returns for each of 25 assets, assuming a multivariate normal distribution with known mean and covariance matrix.

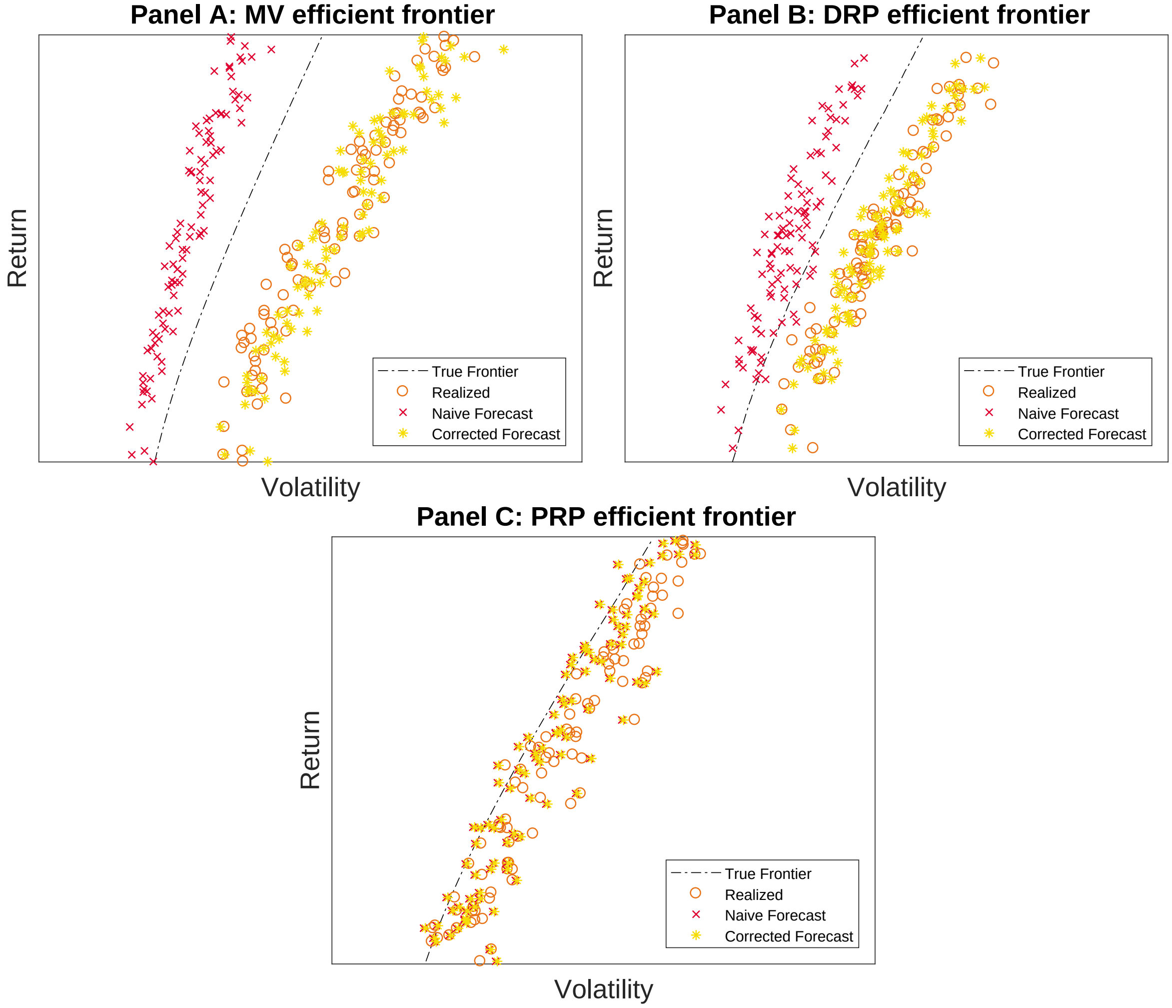

Figure 2. Second-Order Risk Bias under Risk Mitigation Methodologies
The figure provides the second-order risk bias under the application of various estimation risk mitigation methodologies for various asset allocation strategies: the equally weighted portfolio ($1/N$), a random portfolio (Rand.), the minimum variance (MV), the inverse variance ($1/V$), the diversified risk parity (DRP), and the principal risk parity portfolio (PRP). The SOR bias is calculated as the standard deviation of the bias statistic in (37) over a ten-year time period, from April 2007 to March 2017. The bias statistic is calculated as of every day based on a rolling window of 30 daily returns. The asset universe considered consists in the ten industry portfolios provided by the CRSP (Center for Research in Security Prices) and described in Table I. The SOR bias is provided without (Sample) and with the application of an estimation risk mitigation methodology: either the bootstrap, the linear shrinkage, or the eigenvalue adjusting methodology, which were described in Section C.

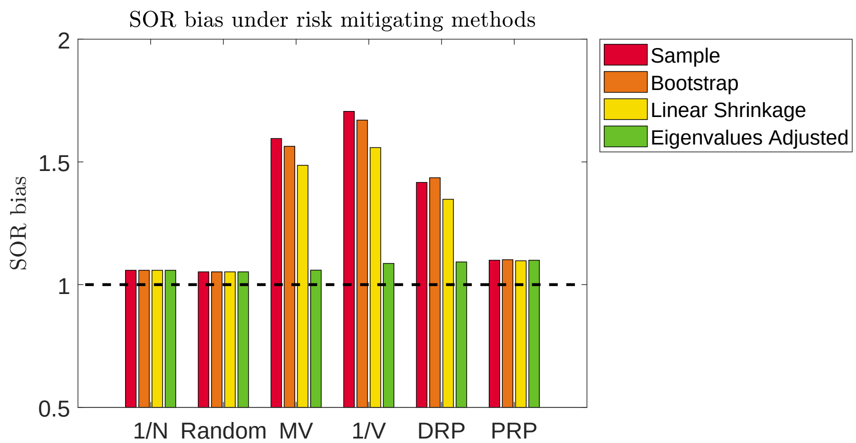

Figure 3. Second-Order Risk Bias Over Time

The figure shows the second-order risk bias over time for various asset allocation strategies and risk mitigation methodologies (as described in Section C. Each panel represents the SOR bias over time for an individual portfolio strategy both with and without the application of the various risk mitigation methodologies. SOR bias is calculated via a rolling window of 30 daily returns. Portfolios are constructed out of the ten industry portfolios described in Table I. Data used range from April 2007 to March 2017.

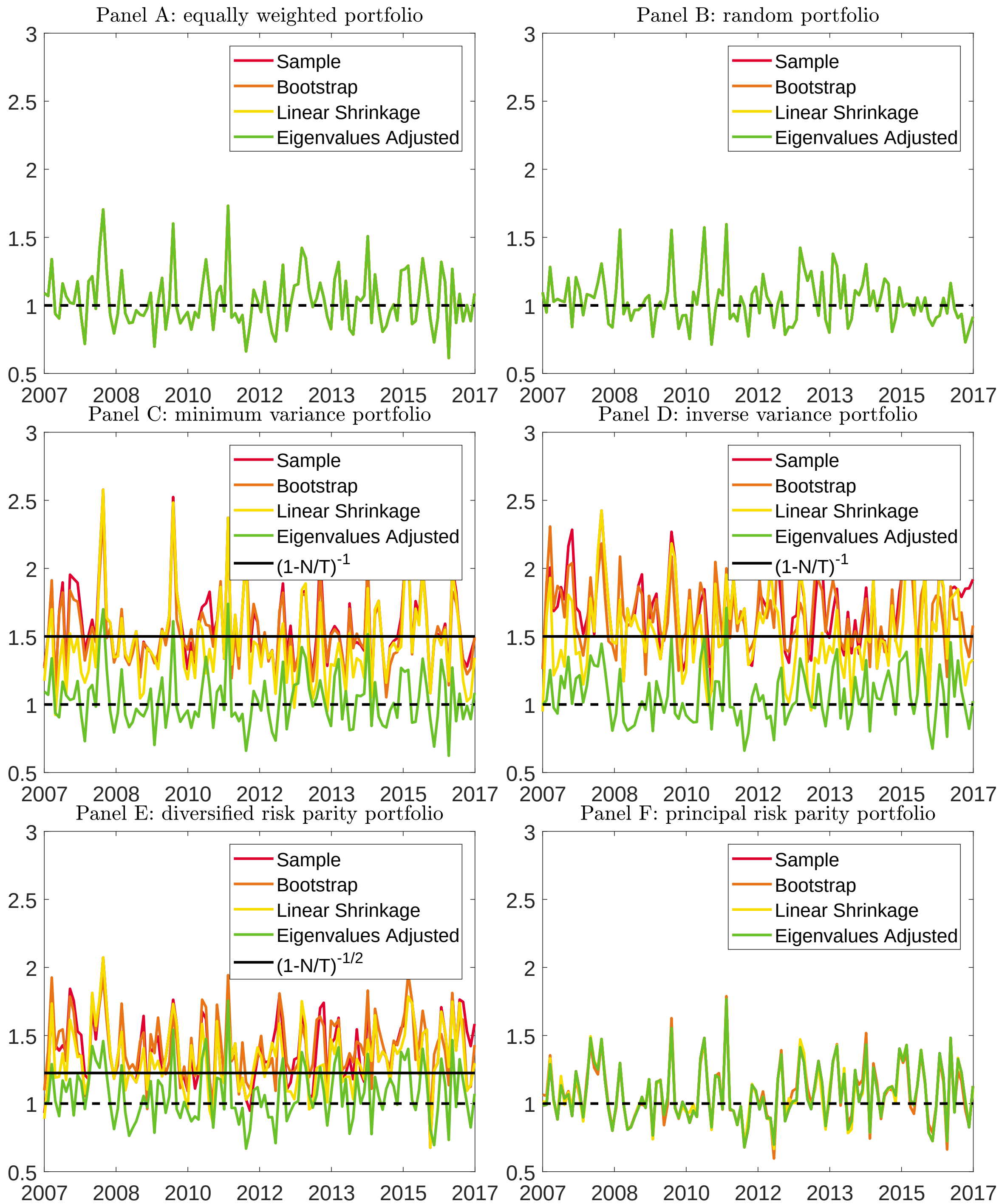

Figure 4. Correcting Second-Order Risk Bias for Varying T/N Ratios

The figure shows the second-order risk bias (calculated via SOR bias statistic) for the MV portfolio, the DRP portfolio and the PRP portfolio under varying ratios of number of monthly returns (T) considered in-sample over number of industry portfolios (N). The number of industry portfolios is kept constant within each panel. For Panel A, a ratio $T/N = 3$ corresponds to the case of 5 industry portfolios and 15 monthly in-sample observations. Panel B displays the SOR bias constructed out of 10 industry portfolios, Panel C with 30 industry portfolios, and Panel D reports the SOR bias based on 48 industry portfolios. Additionally, for MV and DRP strategies, the corrected SOR bias, using estimators from equations (7), and (19) are reported. Results are derived using the 10 industry portfolios described in Table I and analogous industry portfolios (5, 30, and 48) as consistently provided by CRSP (Center for Research in Security Prices). For more details please visit the website: $http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data\_library.html$. Results are generated using data ranging from April 2007 to March 2017.

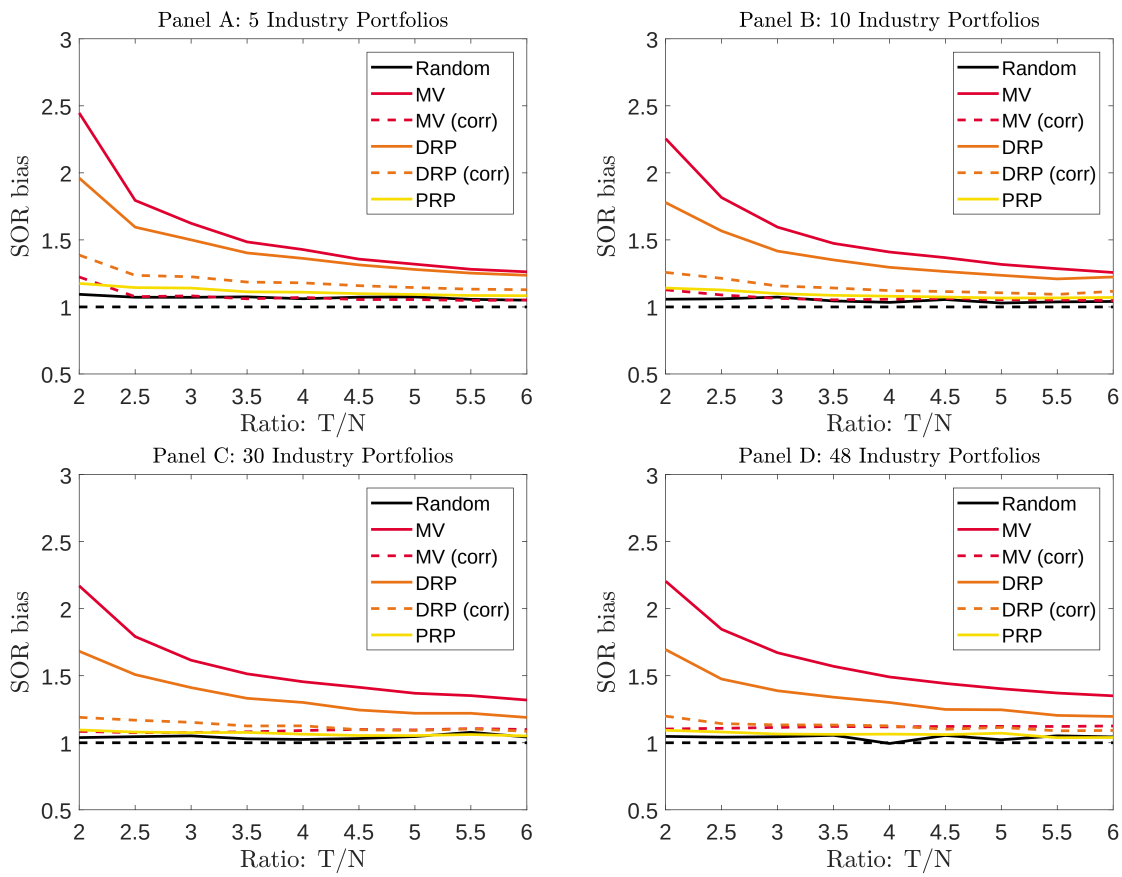

Figure 5. SOR Bias under Risk Mitigation Methodologies for Varying T/N Ratios The figure shows the second-order risk bias (calculated via SOR bias statistic) for the MV portfolio, the DRP portfolio and the PRP portfolio under varying ratios of the number of daily returns (T) considered in-sample to the number of industry portfolios (N). The number of industry portfolios is kept constant at 10. Consequently, a ratio $T/N = 3$ corresponds to the case of 10 industry portfolios and 30 daily in-sample observations. Panel A displays the SOR bias of strategies constructed on the sample covariance matrix. Panel B, C, and D reports the SOR bias conditional on strategies being derived from the corresponding estimator of the covariance matrix: bootstrapping for panel B, linear shrinkage for panel C, and eigenvalue-adjustment for panel D. Results are derived with the asset universe described in Table I and results are generated using data ranging from April 2007 to March 2017.